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Coexistent Effects in Quantum Mechanics†

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Abstract

Effects are defined in this paper as observable changes in the state of a macrosystem, which are caused by interaction with a microsystem. These effects are the starting point of Ludwig's axiomatic foundation of quantum theory. In this theory the concept of commensurability is developed by considering effects which can be caused together, by one single microsystem. Such effects are called coexistent. It is shown that in ordinary quantum mechanics the formal definition of coexistence and the corresponding postulates given by Ludwig are consistent with the dynamics of interaction processes leading to effects.

1. *Introduction*

In recent publications (Ludwig, 1964, 1967a, 1967b) G. Ludwig has given a new axiomatic foundation of quantum mechanics and more general theories. This foundation is not based on J. von Neumann's measuring process or one of its current modifications, thereby circumventing well-known difficulties arising in this connection. The experimental basis consists of some class of real processes induced by microsystems and resulting in observable effects. An effect is defined here as any observable change caused in the state of a macrosystem by a microsystem. The occurrence rate of the effect in long series of experiments under identical macroscopic conditions is $\mu(V,F)$. V is the symbol for the macroscopic conditions under which the microsystems are prepared. F is the symbol for the effect, including the macroscopic conditions of the process resulting in the effect. The determination of $\mu(V,F)$ is reproducible. It is sufficient to reestablish the proper macroscopic conditions. The number $\mu(V,F)$, together with the specification of the macroscopic conditions, should be the only raw material from which the mathematical language expressing properties of microsystems is to be extracted. G. Ludwig's axiomatic foundation of quantum mechanics is adequate for this program.

Effects F and F' are said to be equivalent, if $\mu(V,F) = \mu(V,F')$ holds with arbitrary V, and in analogy V and V' are equivalent, if $\mu(V,F) = \mu(V',F)$ with arbitrary F. In the course of the axiomatic foundation the classes $f(V)$ and $g(F)$ of equivalent V and F are represented by vectors $\mathbf{V}$ in a real Banach space B, and vectors $\mathbf{F}$ in its conjugate B', respectively.

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The definition of B stems from the fact that it is possible to mix the objects resulting from procedures V_1 and V_2 in some proportion $\lambda/1-\lambda$, which yields $\mu(V,F)=\lambda\mu(V_1,F)+(1-\lambda)\mu(V_2,F)$. The class of V is represented by $\mathbf{V}=\lambda\mathbf{V}_1+(1-\lambda)\mathbf{V}_2$. B is defined by allowing real finite coefficients in

$$X=\sum_{i=1}^{n} a_i \mathbf{V}_i$$

and introducing the norm

$$\|X\| = \sup_F |\mu(X,F)| = \sup_F \left|\sum_i a_i \mu(V_i,F)\right|$$

By $\mathbf{F}(X)=\mu(X,F)$ (fixed F) the effects become linear functionals on B. For a rigorous construction see Ludwig (1964).

Particular attention is given to axioms concerning commensurability. To clarify this concept one has to look for different effects F_γ ($\gamma=1,2,\ldots,n$) possibly occurring as results of interaction of one single microsystem with one or several macrosystems, and look for the mathematical structure of the set $l=\{\mathbf{F}_1,\mathbf{F}_2,\ldots,\mathbf{F}_n\}$ formed by $g(F_\gamma)\rightarrow\mathbf{F}_\gamma$ (the arrow stands for 'represented by') in B'. This suggests the definition of coexistent sets l in B'. The axioms state: Whenever a coexistent set $l'=\{\mathbf{F}_1',\mathbf{F}_2',\ldots,\mathbf{F}_{n'}'\}$ is given in B' then there are sets $l_\nu=\{F_1^{(\nu)},F_2^{(\nu)},\ldots,F_{m_\nu}^{(\nu)}\}$ ($\nu\in I$, $m_\nu\geqslant n'$) of effects which may be produced by one single microsystem and $g(F_j^{(\nu)})\rightarrow\mathbf{F}_j$ ($\nu\in I, j=1,2,\ldots,n'$).

If quantum mechanics is a realization of the axiomatic system given by G. Ludwig it must be possible to check the axioms by means of quantum dynamics applied to interaction processes of microsystems and macrosystems resulting in effects. This is a check on the dependence of the truth and completeness of the dynamical principles under consideration. The purpose of this paper is to show that the definition of coexistence meets exactly the situation in quantum mechanics, if one takes into consideration only the unitary representation of the time translations.

In Section 2 the definition of coexistence given by G. Ludwig is repeated. Section 3 contains the analysis of the interaction process leading to Theorem 2, which states the equivalence of the two *statements*: (i) $g(F_1)$ and $g(F_2)$ contain pairs of effects $(F_1',F_2')\in(g(F_1),g(F_2))$, which can be produced together by one single microsystem; (ii) if $g(F_1)\rightarrow\mathbf{F}_1$ and $g(F_2)\rightarrow\mathbf{F}_2$ then $\{\mathbf{F}_1,\mathbf{F}_2\}$ is a coexistent set in B'. The proof of Theorem 2 is given in Section 4.

2. *Definition of Coexistence*

If different effects F_γ ($\gamma=1,2,\ldots,n$) occur as a possible result of the interaction of one single microsystem with one or several macrosystems, it is possible to construct another macrosystem, with effects F_γ' ($\gamma=1,2,\ldots,n,\ldots,m$) possibly occurring in such a manner that with arbitrary V, $\mu(V,F_\gamma)=\mu(V,F_\gamma')$ ($\gamma=1,2,\ldots,n$) hold, and F_γ' ($\gamma>n$) counting the combined occurrences of non-occurrences of effects $F_{\gamma'}'$. ($\gamma'<\gamma$). This can be done by adding some electronics to the first macrosystem (Ludwig, 1964). Let $F_\alpha' \dotplus F_\beta'$ symbolize the effect F_ε' of exclusive occurrence of F_α' and F_β', and let $F_\alpha' \circ F_\beta'$ symbolize the effect F_δ' of the combined occurrence, then, with arbitrary V,

$$\mu(V, F_\alpha') + \mu(V, F_\beta') = \mu(V, F_\alpha' \dotplus F_\beta') + 2\mu(V, F_\alpha' \circ F_\beta') \tag{2.1}$$

must hold. If all combinations of occurrences are included in the set $\{F_1', F_2', \ldots, F_m'\}$, it becomes a boolean algebra $\mathfrak{a}$ with the operations '$\dotplus$' and '$\circ$'. From

$$\mu(V, F_\alpha') + \mu(V, F_\beta') = \mu(V, F_\alpha' \dotplus F_\beta') \qquad \text{if} \quad F_\alpha' \circ F_\beta' = 0 \tag{2.1a}$$

one infers that any V defines an additive measure function on $\mathfrak{a}$. Written in terms of B' by $g(F_\alpha') \to \mathbf{F}_\alpha$, $g(F_\beta') \to \mathbf{F}_\beta$, and $g(F_\alpha' \dotplus F_\beta') \to \mathbf{F}_\varepsilon$ this reads

$$\mathbf{F}_\alpha + \mathbf{F}_\beta = \mathbf{F}_\varepsilon \qquad \text{if} \quad F_\alpha' \circ F_\beta' = 0 \tag{2.1b}$$

Define now by $\tilde{g}(F_\gamma') = \mathbf{F}_\gamma$ if (and only if) $g(F_\gamma') \to \mathbf{F}_\gamma$ a vector measure $\tilde{g}$ on $\mathfrak{a}$. By (2.1a) $\tilde{g}$ is inferred to be additive. This suggests the following definition:

Definition

A finite set l of $\mathbf{F}_\gamma$ $(\gamma = 1, 2, \ldots, n)$ representing classes of effects in B' is coexistent if and only if there is a boolean algebra $\mathfrak{a}$ with an additive vector measure $\tilde{g}$ on $\mathfrak{a}$, such that l is contained in the range of $\tilde{g}$.

To connect this with the following analysis of interaction processes, we use a theorem, which is proved by Ludwig (1967a).

Theorem 1

$\mathbf{F}$, $\mathbf{G}$, representing classes of effects in B', form a coexistent set if and only if there are three vectors $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, with

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2, \qquad \mathbf{G} = \mathbf{F}_1 + \mathbf{F}_3,$$

and $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$ represent classes of effects in B'.

3. *Statement of Objective*

In quantum mechanics B is the Banach space of Hermitean trace class operators of a Hilbert space $\mathfrak{H}_1$. The convex set of statistical operators $W_1 \geqslant 0$, $\operatorname{tr} W_1 = 1$ is the subset of operators in B representing preparation procedures, the convex set of operators F, $F = F^+$, $0 \leqslant F \leqslant 1$ is the subset of operators in B' representing the effects, and $\mu(W_1, F) = \operatorname{tr}(W_1 F)$. We shall refer hereafter to $\mathfrak{H}_1$ as the Hilbert space of the microsystem, and refer to $\mathfrak{H}_2$ as the Hilbert space of the macrosystem. Then $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ is the Hilbert space of the coupled systems. A necessary condition for the possibility of interaction processes translating a statistical operator W_i at time $t = 0$

into W_f at time $t = t_1$ in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ is the existence of a unitary operator U with $W_f = UW_i U^+$. It is only this dynamical principle that we shall consider here.

The guiding idea to describe processes resulting in effects is as follows (Hellwig, 1967). At the beginning of the interaction process with the microsystem the macrosystem is in state of metastable equilibrium. Any change in state of the macrosystem caused by the microsystem must have occurred after a finite time τ. In this respect it is irrelevant whether the microsystem is absorbed by the macrosystem or not. If absorbed the microsystem has lost its surplus energy (or any other properties which may trigger the macrosystem) by multiple scattering and, in a certain sense, is in equilibrium with the macrosystem. If not absorbed the microsystem has left the macrosystem. In any case the macrosystem may be regarded macroscopically as the same before and after the interaction. To ascertain a possibly occurring effect to be caused by the microsystem, τ must be short in comparison to the mean lifetime τ_0 of the metastable equilibrium.

Denote by W_i the statistical operator of the coupled system at the time $t = 0$, when the interaction process begins. Furthermore let W_f be the statistical operator at time $t = t_B > \tau$, but $t_B \ll \tau_0$, in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. When investigating only effects corresponding to equilibrium states of the macrosystem we do not need to fix t_B exactly in a series of experiments. Let $Q^2 = Q$ be the projection in $\mathfrak{H}_2$ which corresponds to the macroscopically observable property of the macrosystem belonging to the effect $\boldsymbol{F}$. The rate of occurrence is

$$\mu(V, \boldsymbol{F}) = \operatorname{tr}(W_f(1 \otimes Q))$$

If the microsystem is absorbed, clearly $1 \otimes Q$ has to be replaced by a more general $\mathbf{Q}$, $\mathbf{Q}^2 = \mathbf{Q}$ in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. However, the difference is mere formal play, since the impurity in a macrosystem caused by a single atom does not alter macroscopically observable properties. Let S be the unitary transformation responsible for the temporal translation from $t = 0$ to $t = t_B$ of the coupled ensemble in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$, then

$$\mu(V, \boldsymbol{F}) = \operatorname{tr}(SW_i S^+(1 \otimes Q))$$

W_i has the form $W_1 \otimes W_M$, where W_1 is the statistical operator corresponding to the preparation procedure V. W_M indicates when the macrosystem is in metastable equilibrium. Hence

$$\mu(V, \boldsymbol{F}) = \operatorname{tr}(S(W_1 \otimes W_M) S^+(1 \otimes Q))$$

It is easy to show that there is one, and only one, linear symmetric operator F, $0 \leqslant F \leqslant 1$ in $\mathfrak{H}_1$ with the property that for any W_1 in $\mathfrak{H}_1$ the following result holds

$$\operatorname{tr}(S(W_1 \otimes W_M) S^+(1 \otimes Q)) = \operatorname{tr}(W_1 F)$$

That is

$$\mu(V, \boldsymbol{F}) = \operatorname{tr}(W_1 F) \tag{3.1}$$

The operator F is determined by the bilinear form

$$(\varphi, F\chi) = \sum_j w_j(\varphi\psi_j, S^+(1 \otimes Q) S\chi\psi_j) \tag{3.2}$$

defined for any two $\varphi, \chi \in \mathfrak{H}_1$. The w_j and ψ_j are due to a spectral representation $W_M = \sum_j w_j P_{\psi_j}$. $(\varphi, F\chi)$ is insensitive to rotations in eigenspaces of W_M.

The rate $\mu(V, F)$ of occurrence of F, at the time $t = t_B$, is identified here by the value of $\operatorname{tr}(W_1 F)$ at the time $t = 0$. When we know the unitary representation of temporal translations U_t in $\mathfrak{H}_1$ for the microsystems it is a straightforward matter to identify these values at the same time ($t = t_B$) or in some other way. For the consideration below we need the structure of connections (2.1) and (3.1) only.

If there are several different outcomes of such processes possible which are macroscopically observable the corresponding projection operators Q_γ ($\gamma = 1, 2, \ldots, n$) must commute. For reasons of simplicity the discussion will be restricted to the two properties P and Q. Then the following theorem holds:

Theorem 2

Let two linear symmetric operators F and G in $\mathfrak{H}_1$ with

$$0 \leqslant F \leqslant 1; \qquad 0 \leqslant G \leqslant 1 \tag{3.3}$$

be given. Now choose two different projection operators Q and P in $\mathfrak{H}_2$ with

$$[Q, P] = 0 \tag{3.4}$$

and

$$1 - Q - P + QP \neq 0 \tag{3.5}$$

Finally define a statistical operator W_2, by choosing N mutual orthogonal vectors ψ_j, and N positive numbers w_j with $\sum_j w_j = 1$.

A unitary operator S in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ yielding

$$(\varphi, F\chi) = \sum_j w_j(\varphi\psi_j, S^+(1 \otimes Q) S\chi\psi_j) \tag{3.6}$$

$$(\varphi, G\chi) = \sum_j w_j(\varphi\psi_j, S^+(1 \otimes P) S\chi\psi_j) \tag{3.7}$$

for any two vectors φ, $\chi \in \mathfrak{H}_1$ exists if and only if there are three linear operators F_α ($\alpha = 1, 2, 3$) with

$$0 \leqslant F_\alpha; \qquad 0 \leqslant F_1 + F_2 + F_3 \leqslant 1 \tag{3.8}$$

and

$$F = F_1 + F_2; \qquad G = F_1 + F_3 \tag{3.9}$$

This shows that if two effects are not represented by operators F, G, which form a coexistent set of operators in B', they can in no way be generated as result of an interaction process of a macrosystem with one single microsystem, because the desired process contradicts a general dynamical principle. This process is possible at least in principle if and only if $\{F, G\}$ form a coexistent set in B'.

Assumption (3.5) looks somewhat artificial. That it is necessary in general one infers from the following corollary:

Corollary

Assume equations (3.6) and (3.7) to hold with $Q+P-QP=1$, then $F_1+F_2+F_3=1$.

If conversely equations (3.8) and (3.9) hold with $F_1+F_2+F_3=1$ it is possible to choose $Q+P-QP$ equal to 1 or less than 1.

4. *Proof of Theorem* 2

a. *Necessity*

If we write

$$Q_1 = QP, \qquad Q_2 = Q - QP, \qquad Q_3 = P - QP \tag{4.1}$$

then $Q_\alpha{}^2 = Q_\alpha$. Define F_α $(\alpha = 1,2,3)$ by

$$(\varphi, F_\alpha \chi) = \sum_j w_j(\varphi\psi_j, S^+(1 \otimes Q_\alpha) S\chi\psi_j) \tag{4.2}$$

Taking (3.6) into consideration, from $S^+((1 \otimes Q_1)+(1 \otimes Q_2))S = S^+(1 \otimes Q)S$ we get at once $(\varphi, (F_1+F_2)\chi) = (\varphi, F\chi)$ for any two $\varphi, \chi \in \mathfrak{H}_1$, hence $F = F_1 + F_2$. Analogously we show that $G = F_1 + F_3$. Further $0 \leqslant F_\alpha$ $(\alpha = 1,2,3)$. From

$$Z = \sum_{\alpha=1}^{3} Q$$

it follows that

$$\left(\varphi, \sum_{\alpha=1}^{3} F_\alpha \varphi\right) = \sum_j w_j(\varphi\psi_j, S^+(1 \otimes Z) S\varphi\psi_j) \leqslant 1$$

which completes the proof of necessity.

b. *Sufficiency*

Extend the number of mutual orthogonal vectors ψ_j by N other vectors ψ_{N+j} $(j = 1,2,3,\ldots,N)$ resulting in $2N$ normed and mutual orthogonal vectors. If we denote by $\mathfrak{A}_j$ the N subspaces of $\mathfrak{H}_2$, spanned by $\{\psi_j, \psi_{N+j}\}$, fixed N and j, we have

$$\mathfrak{H}_2 = \left\{\sum_{j=1}^{N}{}_{\oplus}\, \mathfrak{A}_j\right\} \oplus \mathfrak{R} \tag{4.3}$$

and

$$\mathfrak{H}_1 \otimes \mathfrak{H}_2 = \left\{\sum_{j=1}^{N}{}_{\oplus}\, (\mathfrak{H}_1 \otimes \mathfrak{A}_j)\right\} \oplus (\mathfrak{H}_1 \otimes \mathfrak{R}) \tag{4.4}$$

Put

$$F_4 = 1 - \sum_{\alpha=1}^{3} F_\alpha \tag{4.5}$$

The conditions of a well-known theorem (Riesz & Nagy, 1960) then apply: Every finite or infinite sequence $\{F_n\}$ of bounded self-adjoint transformations in the Hilbert space $\mathfrak{H}_1$ such that

$$F_n \geqslant 0, \qquad \sum_n F_n = 1 \tag{4.6}$$

can be represented in the form

$$F_n = \operatorname{pr} \mathbf{E}_n \qquad (n = 1, 2, 3, \ldots)$$

where $\{\mathbf{E}_n\}$ is a sequence of projections of an extension space $\boldsymbol{\mathfrak{H}}$ for which

$$\mathbf{E}_n . \mathbf{E}_m = 0 \qquad (n \neq m) \tag{4.7}$$

$$\sum_n \mathbf{E}_n = 1 \qquad (n \neq m) \tag{4.8}$$

$\operatorname{pr} \mathbf{E}_n$ means the following: Let $\mathbf{P}$ be the projection in $\boldsymbol{\mathfrak{H}}$ onto $\mathfrak{H}_1$, then

$$\mathbf{P}\mathbf{E}_n \mathbf{P} = \operatorname{pr} \mathbf{E}_n \tag{4.9}$$

By the following method, N arbitrary extension spaces $\boldsymbol{\mathfrak{H}}_j$ of $\mathfrak{H}_1$ ($j = 1, 2, 3, \ldots, N$) formed in the sense of the theorem quoted are embedded in $\mathfrak{H}_1 \otimes \mathfrak{A}_j$. First identify $\varphi \in \mathfrak{H}_1 \subseteq \boldsymbol{\mathfrak{H}}_j$ by $\varphi\psi_j \in \mathfrak{H}_1 \otimes \mathfrak{A}_j$. Let $\{\Omega_k\}$ denote a fixed complete orthonormal system in $\boldsymbol{\mathfrak{H}}_j \ominus \mathfrak{H}_1$ and $\{\chi_\nu\}$ a fixed complete orthonormal system in $\mathfrak{H}_1$, then identify

$$\phi = \sum_{k=1}^{M} c_k \Omega_k \in \boldsymbol{\mathfrak{H}}_j \ominus \mathfrak{H}_1$$

by $\mathrm{f}\psi_{N+j} \in \mathfrak{H}_1 \otimes \mathfrak{A}_j$ where

$$\mathrm{f} = \sum_{k=1}^{M} c_k \chi_{5k}$$

Clearly

$$(\phi, \phi') = (\mathrm{f}, \mathrm{f}')(\psi_{N+j}, \psi_{N+j}) = (\mathrm{f}, \mathrm{f}') = \sum_{k=1}^{M} c_k^* c_k' \tag{4.10}$$

Then limit elements of converging infinite sequences $\{\phi_\nu\}$ in $\boldsymbol{\mathfrak{H}}_j \ominus \mathfrak{H}_1$ may be identified by the limit elements of $\mathrm{f}_\nu \psi_{N+j} \in \mathfrak{H}_1 \otimes \mathfrak{A}_j$. The unique decomposition of any $\Psi \in \boldsymbol{\mathfrak{H}}_j$ in $\Psi = \sum_k b_k \Omega_k + \psi$ with $\sum_k b_k \Omega_k \in \boldsymbol{\mathfrak{H}}_j \ominus \mathfrak{H}_1$ and $\psi \in \mathfrak{H}_1$ guarantees that any element of $\boldsymbol{\mathfrak{H}}_j$ is identified by an element of $\mathfrak{H}_1 \otimes \mathfrak{A}_j$. The projection $\mathbf{P}_j$ corresponding to $\mathbf{P}$ in the theorem quoted is given by

$$\mathbf{P}_j = 1 \otimes P_{\psi_j}$$

Let $\mathbf{T}_\beta^{(j)}$ be the projections onto the subspaces of $(\mathfrak{H}_1 \otimes \mathfrak{A}_j) \ominus \boldsymbol{\mathfrak{H}}_j$ spanned by $\{\chi_{5\nu-\beta}\psi_{N+j}\}$ $(\beta = 1, 2, 3, 4)$. In case of $\dim(\boldsymbol{\mathfrak{H}}_j \ominus \mathfrak{H}_1) = d < \infty$ the projection onto the subspace spanned by the vectors $\{\chi_{5(d+\nu)}\psi_{N+j}\}$ shall be $\mathbf{R}^{(j)}$, if $d=\infty$ then $\mathbf{R}^{(j)}=0$. Denote by $\mathbf{E}_\beta^{(j)}$ ($\beta=1,2,3,4$) the operators with the properties (4.6) through (4.9) in $\mathfrak{H}_j$ then

$$E_\alpha^{(j)}=\mathbf{T}_\alpha^{(j)}+\mathbf{E}_\alpha^{(j)} \qquad (\alpha=1,2,3) \tag{4.11}$$

$$E_4^{(j)}=\mathbf{T}_4^{(j)}+\mathbf{E}_4^{(j)}+\mathbf{R}^{(j)} \tag{4.12}$$

with

$$(E_\beta^{(j)})^2=E_\beta^{(j)},\ E_\beta^{(j)}E_{\beta'}^{(j)}=0 \qquad (\beta\neq\beta') \tag{4.13}$$

$$\sum_{\beta=1}^{4} E_\beta^{(j)}=1_{\mathfrak{H}_1\oplus\mathfrak{A}_j} \tag{4.14}$$

$$\dim E_\beta^{(j)}(\mathfrak{H}_1\otimes\mathfrak{A}_j)=\infty \tag{4.15}$$

$$(1\otimes P_{\psi_j})E_\beta^{(j)}(1\otimes P_{\psi_j})=F_\beta\otimes P_{\psi_j} \tag{4.16}$$

($\beta,\beta'=1,2,3,4$). Considering (4.4), define in $\mathfrak{H}_1\otimes\mathfrak{H}_2$

$$\mathbf{E}_\alpha=\left\{\sum_{j=1}^{N}{}^{\oplus}\ E_\alpha^{(j)}\right\}\oplus 0_{\mathfrak{h}_1\oplus\mathfrak{R}} \qquad (\alpha=1,2,3) \tag{4.17}$$

and

$$\mathbf{E}_4=\left\{\sum_{j=1}^{N}{}^{\oplus}\ E_4^{(j)}\right\}\oplus 1_{\mathfrak{h}_1\oplus\mathfrak{R}} \tag{4.18}$$

The relations (4.13) through (4.16) also hold for $\mathbf{E}_\beta$, if $\mathfrak{H}_1\otimes\mathfrak{A}_j$ is replaced by $\mathfrak{H}_1\otimes\mathfrak{H}_2$. Therefore a unitary operator S in $\mathfrak{H}_1\otimes\mathfrak{H}_2$ with

$$\mathbf{E}_\alpha=S^+(1\otimes Q_\alpha)S \qquad (\alpha=1,2,3) \tag{4.19}$$

$$\mathbf{E}_4=S^+\left(1\otimes\left(1-\sum_{\alpha=1}^{3}Q_\alpha\right)\right)S \tag{4.20}$$

exists. For any j ($j=1,2,\ldots,N$), ($\alpha=1,2,3$), and any two φ, $\chi\in\mathfrak{H}_1$ we have

$$(\varphi\psi_j,S^+(1\otimes Q_\alpha)S\chi\psi_j)=(\varphi\psi_j,\mathbf{E}_\alpha\chi\psi_j)=(\varphi\psi_j,E_\alpha^{(j)}\chi\psi_j)=(\psi,F_\alpha\chi)$$

Also

$$\sum_{j=1}^{N}w_j=1$$

thus

$$(\varphi,F_\alpha\chi)=\sum_{j=1}^{N}w_j(\varphi\psi_j,S^+(1\otimes Q_\alpha)S\chi\psi_j) \tag{4.21}$$

From a consideration of (4.1) and assumption (3.9) the relations (3.6) and (3.7) follow which completes the proof of sufficiency.

For a proof of the first supposition of the corollary note that

$$Q + P - QP = 1$$

implies

$$\sum_{\alpha=1}^{3} Q_\alpha = 1$$

By adding up equations (4.2), one infers $\sum_\alpha F_\alpha = 1$. The proof of the second supposition can be performed in analogy to the proof of sufficiency of the theorem.

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